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USE OF LAGUERRE FILTERS FOR REALISATION OF TIME FUNCTIONS AND DELAY

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ABSTRACT. In this paper is presented a method of realising time functions, delays and delayed integrations by making use of Laguerre filters. Circuit arrangements for realising two classes of Laguerre functions are described. The method has the merit that the adjustments required for obtaining different operations present no difficulty.

1. INTRODUCTION

In a simulator, the components of the original system are replaced by analogous devices having similar performance characteristics. One of the basis of such simulation is the time response. Further, in simulation of process control systems, one is required to realise operations of delay and delayed integration.

It is known that for the indicial response of a system a sectionalised delay line may be used. (Corrington *et al.* 1954). But, for low frequency applications, the construction of such delay line is difficult. For the realisation of time delays, a method employing operational amplifiers has been described (Morill, 1954); this method labours under the disadvantage that it requires a large number of elements and is not adaptable to the realisation of time functions.

2. LAGUERRE FUNCTIONS

The simplest means of generating a function would be to sum a number of easily realisable functions. For this purpose it would be a great advantage if the coefficients could be easily determined. The Laguerre functions, besides being orthogonal, also possess rational Fourier transforms, and are thus eminently suited for the realisation of arbitrary time and frequency functions.

A set of Laguerre functions due to Tricomi is defined as

$$T_n(t) = \sum_{0}^{r=n} \binom{n}{r} \frac{(-t)^r}{r!} = L^{-1} \frac{(p-1)^n}{p^{n+1}} = L^{-1}\, T_n(p) \qquad \ldots \quad (1)$$

Now

$$L^{-1} e^{-\alpha t} T_n(t) = \frac{1}{p+\alpha} \left(\frac{p+\alpha-1}{p+\alpha} \right)^n \qquad \ldots \quad (2)$$

Hence writing $g(p) = \frac{1}{p+\alpha} \Sigma a_n \left(\frac{p+\alpha-1}{p+\alpha}\right)^n$, one has

$$\mathcal{L}^{-1}g(p) = F(t) = e^{-\alpha t} \Sigma a_n T_n(t) \quad \ldots \quad (3)$$

The coefficients a_n are given by the series

$$(p+\alpha)\, g(p) = \Sigma a_n Z^n = F(Z) \quad \ldots \quad (4)$$

where the transformation $Z = \frac{p+\alpha-1}{p+\alpha}$ has been adopted. This relation transforms any circle $|Z| = a$ in the z-plane into a circle on the p-plane which may be made to enclose all the singularities of $(p+\alpha)\, g(p)$ by choosing suitable values of a and α.

We now consider the different functions obtained by giving values to α.

α	Z	Transformation	Radius of convergence of F(z)	Time functions
0	$\frac{p-1}{p}$	Half plane $R(p) < 0$ into the half plane $R() > 1$	$R(Z) < 1$	$\sum \binom{n}{r} \frac{(-t)^r}{r!} = T_n(t)$
$\frac{1}{2}$	$\frac{p-1/2}{p+1/2}$	Half plane $R(p) < 0$ into an area outside the unit circle	Inside the unit circle	$e^{-t/2} \sum \binom{n}{r} \frac{(-t)^r}{r!} = L_n(t)\,(-1)^n$
1	$\frac{p}{p+1}$	Left half-plane of p into the right half plane of Z	$\|Z\| = \frac{1}{2}$ at $Z = \frac{1}{2}$	$e^{-t} \sum \binom{n}{r} \frac{(-t)^r}{r!} = \beta_n(t)$

It is to be noted that only the symmetric function corresponding to $\alpha = 1/2$ permits expansion of $F(Z)$ in powers of Z around $Z = 0$. Therefore even in realising a function in terms of $\beta_n(t)$ the preliminary mathematical steps should employ only the set $L_n(t)$. The conversion is safely effected by noting that

$$\frac{p}{p+1} = 1/2 \left(1 + \frac{p-1}{p+1}\right)$$

3. GENERATION OF ARBITRARY TIME FUNCTIONS

Let $f(t)$ be the impulse response of the system to be simulated and $g(p)$ be the Laplace transform of $f(t)$. The steady value of $f(t)$ is assumed to be zero. If it is not, we consider the function $f(t) - [f(t)]_\infty = F(t)$. We may now write

$$F(t) = \sum^n a_k L_k(t) = e^{-t}\, \Sigma \beta_k t^k$$

which means that it is desired to approximate $F(t)e^t$ by a polynomial in t. The relation between the coefficients a_k and β_k is simple. For the purpose we note that

$$\frac{t^k e^{-t}}{k!} = \mathcal{L}^{-1}\frac{1}{(p+1)^{k+1}} = \mathcal{L}^{-1}\frac{1}{2^k(1+p)}\left(1+\frac{1-p}{1+p}\right)^k$$

$$= 1/2^k \mathcal{L}^{-1}\left[L_0(p)+kL_1(p)+\ldots\ldots\right] \quad \ldots \quad (5)$$

In case $g(p)$ is known analytically or approximately by the expansion in terms of the moments of $f(t)$ we effect the transformation

$$g\left(\frac{Z-1}{Z+1}\right) \cdot (1+Z) = \Sigma a_n Z^n \quad \ldots \quad (6)$$

It is clear that the procedure seeks to find the numerator coefficients on the assumption of a given denominator polynomial $(1+p)^{n+1}$. Besides in a few cases expansions of the arbitrary functions are readily recognisable. For example, one finds easily that

$$J_0(2\sqrt{x\,t}) = e^{-t/k}\sum_{0}^{n=\infty}\frac{1}{n!}\left(\frac{t}{k}\right)^n L_n(kx)(-1)^n \quad \ldots \quad (7)$$

$$\frac{erf(\sqrt{t})}{\sqrt{\pi}} = e^{-t}\sum_{0}^{n=\infty}\frac{(n+\frac{1}{2})}{n\,!}(-1)^{n-1}L_{n-1}(t) \quad \ldots \quad (8)$$

For realisation of a square pulse of duration T, one has only to evaluate

$$a_n = \int_0^T L_n(t)\,dt.$$

We give below expansions of some simple functions in terms of Laguerre function.

1. Delay Operator :

$$e^{-pT} = \frac{1}{p+1}\sum_0^\infty (-1)^n L_n(T)\frac{(1-p)^n}{(1+p)^n}$$

2. Square wave of duration T :

$$\frac{1-e^{-pT}}{p} = \frac{1}{p+1}\sum_0^\infty\left(\frac{1-p}{1+p}\right)^n\int_0^T L_n(t)dt$$

3. Integrator :

$$\frac{1}{p} = \frac{1}{p+1}\sum_0\left(\frac{1-p}{1+p}\right)$$

4. Delayed Integration :

$$\frac{1}{p} - \frac{1-e^{-pT}}{p} = \frac{1}{p+1}\sum_{0}^{n=\alpha}\left(\frac{1-p}{1+p}\right)^n \left[1-\int_0^T L_n(t)dt\right]$$

To realise an arbitrary function it is only necessary to add different order Laguerre function multiplied by appropriate coefficients in proper phase. The values of the co-efficients for generating a square pulse is presented in Table I. It will be noted that the operation of delayed integration can be realised more easily by subtracting the output of a system realising a square pulse of duration T from the output of the conventional integrator when both are excited by the same input. It is only necessary to adjust the time constant of the integrator to equal the duration T.

It is to be noted that different order Laguerre functions have been realised by CR networks. The time scale is therefore normalised with respect to the CR time constant of the networks.

4. CIRCUIT FOR GENERATION OF LAGUERRE FUNCTIONS

The Laplace transform of the Laguerre functions of the two sets corresponding to $\alpha = 1$ and $\alpha = \frac{1}{2}$ can be written as

$$e^{-bt}T_n(bt) = L^{-1}\frac{1}{p+b}\left(\frac{p}{p+b}\right)^n$$

$$(-1)^n e^{-bt}T_n(2bt) = L^{-1}\frac{1}{p+b}\left(\frac{b-p}{b+p}\right)^n$$

TABLE I

For square pulses of duration T.

Duration	Coefficients of different Laguerre functions										
T	L_0	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
0 25	0 442	.336	.248	.174	.114	065	026	.006	−.023	−.048	− 051
0.50	0 787	426	181	.021	− 072	− 120	−.134	−.129	− 107	−.083	− 051
1 00	1 264	.207	−.207	− 283	−.207	− 087	.024	.091	.133	.198	.137
1.50	1.557	−.215	− 454	− 215	.048	186	199	.129	053	−.074	− 097
2.00	1.729	−.647	−.436	.079	278	.227	.041	− 102	−.170	−.147	− 079
2.50	1.836	−1.015	−.216	.353	.207	.011	−.188	− 209	−.105	.031	.122
3.00	1.900	−1 303	.108	.490	.108	−.228	−.250	− 074	.018	.183	.172
3.50	1.940	−1 517	.460	450	−.157	−.333	−.111	.144	.218	.119	−.038
4.00	1.963	−1 670	.791	.283	−.381	− 232	.051	.324	.069	−.011	−.245

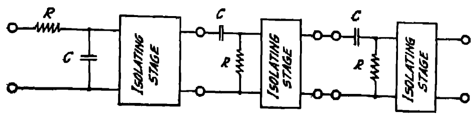

Fig. 1(a). Schematic circuit for realising Laguerre functions corresponding to $\alpha = 1$

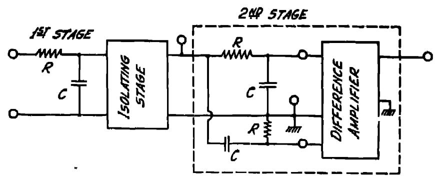

Fig. 1 (b). Corresponding to $\alpha = \frac{1}{2}$

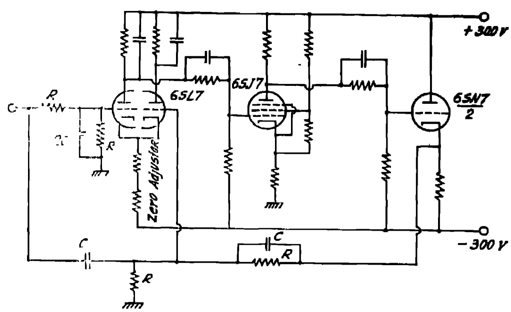

Fig. 1(c). Circuit arrangement for realising Laguerre function corresponding to $\alpha = \frac{1}{2}$

Schematic circuits for realising the Laguerre functions corresponding to $\alpha = 1$ and $\alpha = 1/2$ are shown in figure 1(a) and 1(b) respectively. In figure 1(b) the output from the first stage having the transfer function $1/pCR+1$ is fed through an isolating cathode follower into networks having transfer functions $1/pCR+1$ and $pCR/pCR+1$. The output of the networks are fed into the two inputs of a unity gain difference amplifier The transfer function of the stage is therefore $\dfrac{1-pCR}{1+pCR}$. The practical circuit arrangement is shown in figure 1(c).

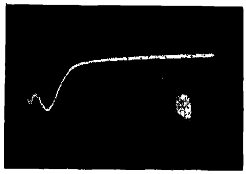

Fig. 2 (*a*). Photographs of step response of the different order Laguerre functions corresponding to $\alpha = \frac{1}{2}$ 3rd order;

Fig. 2 (*b*). 4th order;

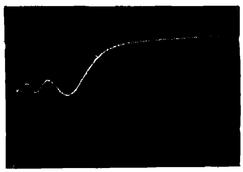

Fig. 2 (*c*). 6th order;

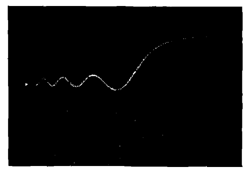

Fig. 2 (*d*). 8th order;

Fig. 2. (*e*) 10th order

Photographs of the step response of the different order Laguerre functions is shown in figure 2.

5. EXAMPLES TO ILLUSTRATE FUNCTION GENERATION

In the system built there are ten Laguerre filters (with $\alpha = \frac{1}{2}$) connected in cascade. The output from each filter is fed into a potentiometer for coefficient multiplication. The addition of the Laguerre functions in proper phase is done

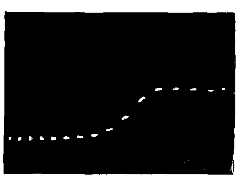

Fig. 3(*a*). Photograph of the step response of a delay function synthesised with Laguerre filters with a delay of 0.6 sec.

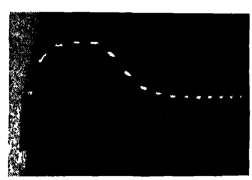

Fig. 3(*b*). Photograph of square pulse

by feeding the outputs from the different potentiometers into the proper terminals of a unity gain difference amplifier.

Photographs of step response of a delay function with a delay of 0.6 sec. and a square wave function of duration 0.6 sec. are shown respectively in figures 3(a) and 3(b).

6. CONCLUSION

In the study of process control systems with the help of a differential analyser the Laguerre filters can be used to realise time delays with advantage. The filters can also be used for simulation of the components of the original system.

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